CHAPTER
EIGHT

VIII. FINITE ELEMENT METHOD FOR ORDINARY DIFFERENTIAL EQUATION

The finite element (FE) method is more versatile than the finite difference method because of the FE freedom of selecting an arbitrary distribution of mesh points. It has several advantages over some other numerical methods: (1) it can deal with irregular geometries, (2) its accuracy is generally higher than in the finite difference method, (3) the higher order approximations can be easily obtained, (4) the treatment of boundary conditions is easier, and (5) the derivation of FE approximation is sometiomes easier. However, there some drawbacks: (1) calculation of the coefficients for the approximating equations is longer, (2) the matrix of the discretized system is irregular, (3) must use the direct methods of matrix inversion in the solution of resulting linear algebraic systems of equations. The finite element method is widely used in heat transfer, neutron transport, neutron diffusion, and fluid flow.

VIII.1. Finite Element method for the diffusion equation in 1D slab geometry

We start from the neutron diffusion equation in 1D slab geometry, with vacoom boundary conditions:

$$-\frac{d}{dx}D(x)\frac{d}{dx}\Phi(x) + \Sigma_a(x)\Phi(x) = Q(x), \; x \in [0, H] \qquad \text{(VIII. 1)}$$

$$\text{B.C.} \quad \Phi(0) = 0$$

$$\left.\frac{d\Phi}{dx}\right|_{x = H/2} = 0$$

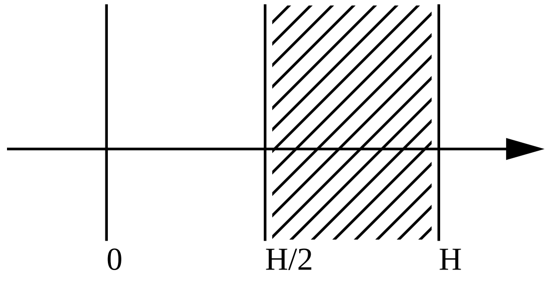


FIGURE VIII.1. The one dimensional slab geometry

We now impose the following spatial mesh that divides the slab into "cells":

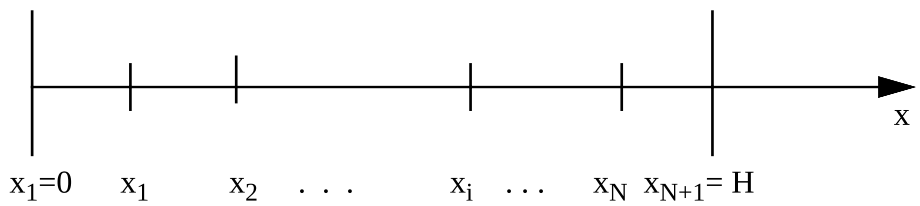


FIGURE VIII.2. Cell-edged spatial mesh

Now we will expand $\Phi(x)$ in terms of the finite elements basis functions

$$\Phi(x) = \sum_{j=1}^{N+1} \Phi_j \Psi_j(x) \qquad \text{(VIII. 2)}$$

where Φ_j - are the cell-edged fluxes, and

$$\Psi_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{h_{j-1}} & , x_{j-1} \le x \le x_j \\ \dfrac{x_{j+1} - x}{h_j} & , x_j \le x \le x_{j+1} \\ 0 & , otherwise \end{cases} \qquad \text{(VIII. 3)}$$

where $h_j = x_{j+1} - x_j$.

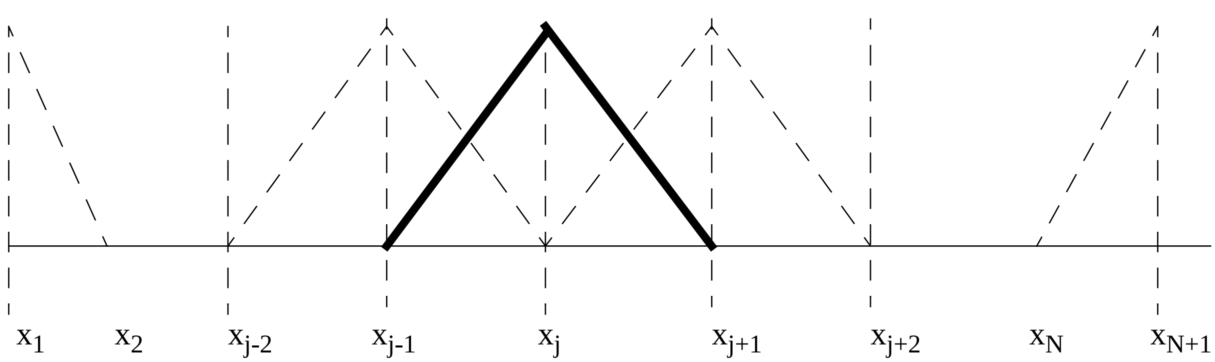


FIGURE VIII.3. Finite-element basis functions

Using Eqs. (2) and (3) we can get

$$\Phi(x) = \Phi_j \Psi_j(x) + \Phi_{j+1} \Psi_{j+1}(x) \qquad , x_j \le x \le x_{j+1} \qquad \text{(VIII. 4)}$$

or

$$\Phi(x) = \frac{1}{h_j}[\Phi_j \cdot (x_{j+1} - x) + \Phi_{j+1}(x - x_j)] \quad , x_j \le x \le x_{j+1} \qquad \text{(VIII. 5)}$$

and $\Phi(x_j) = \Phi_j, \quad j = 1, 2, \ldots, N+1$. If we multiply Eq. (1) by $\Psi_i(x)$ and integrate over $x \le (0, H)$ we have

$$\langle \Psi_i, -\frac{d}{dx} D(x) \frac{d}{dx} \Phi(x) \rangle + \langle \Psi_i, \Sigma_a(x) \Phi(x) \rangle = \langle \Psi_i, Q(x) \rangle \qquad \text{(VIII. 6)}$$

(1) (2) (3)

where

$$\langle f, g \rangle = \int_0^H f(x) g(x) dx \qquad \text{(VIII. 7)}$$

Now we can determine each term in Eq. (6):

$$(1) = -\left(D(x)\frac{d\Phi}{dx}\right)\bigg|_H \delta_{i, N+1} - \left(-D(x)\frac{d\Phi}{dx}\right)\bigg|_0 \delta_{i,1} + \frac{D_{i-1}}{h_{i-1}}(\Phi_i - \Phi_{i-1}) + \frac{D_i}{h_i}(\Phi_i - \Phi_{i+1})$$

$$(2) = \frac{\Sigma_{a, i-1} h_{i-1}}{6} \Phi_{i-1} + \frac{1}{3}(\Sigma_{a, i-1} h_{i-1} + \Sigma_{a, i} h_i) \Phi_i + \frac{\Sigma_i h_i}{6} \Phi_{i+1}$$

$$(3) = \frac{1}{6k}[\nu\Sigma_{f, i-1} \Phi_{i-1} + 2(\nu\Sigma_{f, i-1} h_{i-1} + \nu\Sigma_{f, i} h_i) \Phi_i + \nu\Sigma_{f, i} h_i \Phi_{i+1}$$

an write the final set of equations as

$$a_i \Phi_{i-1} + b_i \Phi_i + c_i \Phi_{i+1} = Q_i \quad i = 2, 3, \ldots, N+1 \qquad \text{(VIII. 8)}$$

$$\text{B.C.} \quad \Phi(0) = 0 \Rightarrow \Phi_1 = 0$$

$$J\left(\frac{H}{2}\right) = 0$$

We can now determine the expressions for m; and q_i in the algorithm

$$\Phi_i = m_i - q_i\Phi_{i+1} \tag{VIII. 9}$$

At $x = \frac{H}{2}$ write two equations

$$\Phi_{N+1} = m_{N+1} - q_{n+1}\Phi_{N+2} = m_{N+1} - q_{N+1} \cdot \Phi_N \tag{VIII. 10}$$

Due to symmetry

$$\Phi_{N+2} = \Phi_N \tag{VIII. 11}$$

(VIII. 12)

$$\Phi_N = m_n - q_N\Phi_{N+1} \tag{VIII. 13}$$

and solve these two equations (i.e. eliminate Φ_N)

$$\Phi_{N+1} = \frac{m_{N+1} - q_{N+1}m_N}{1 - a_{N+1}q_N} \tag{VIII. 14}$$

Now, the same algorithm as in the finite difference solution can be used.

VIII.2. Derivation of the Finite Element Coefficients

<u>Derivation of the first term from Eq. (6)</u>

$$(1) = -\int_0^H dx\Psi_i(x)\frac{d}{dx}D(x)\frac{d}{dx}\Phi(x)$$

Use

$$\frac{d}{dx}\left[\Psi_i(x)\left(D(x)\frac{d}{dx}\Phi(x)\right)\right] = \left(\frac{d}{dx}\Psi_i(x)\right) \cdot \left(D(x)\frac{d}{dx}\Phi(x)\right) + \Psi_i(x)\frac{d}{dx}\left(D(x)\frac{d}{dx}\Phi(x)\right)$$

and

$$-\Psi_i(x)\frac{d}{dx}D(x)\frac{d}{dx}\Phi(x) = -\int_0^H dx\left[\frac{d}{dx}\left(\Psi_i(x)D(x)\frac{d}{dx}\Phi(x)\right)\right] + \frac{d}{dx}\Psi_i(x) \cdot \left(D(x)\frac{d}{dx}\Phi(x)\right)$$

Then

$$(1) = -\int_0^H dx\frac{d}{dx}\left(\Psi_i(x)D(x)\frac{d}{dx}\Phi(x)\right) + \int_0^H dx\frac{d}{dx}\Psi_i(x)\cdot D(x)\frac{d}{dx}\Phi(x)$$

$$= -\Psi_i(x)D(x)\frac{d}{dx}\Phi(x)\big|_H + \Psi_i(x)D(x)\frac{d}{dx}\Phi(x)\big|_0 + \int_0^H dx\frac{d}{dx}\Psi_i(x)D(x)\frac{d}{dx}\sum_{j=1}^{N+1}\Phi_j\Psi_j(x)$$

Use

$$\frac{d}{dx}\Psi_j(x) = \begin{cases} \frac{1}{h_{j-1}} & , x_{j-1} \le x \le x_j \\ -\frac{1}{h_j} & , x_j \le x \le x_{j+1} \end{cases}$$

$$(1) = -\Psi_{N+1}(H)D(x)\frac{d}{dx}\Phi(x)\big|_H\delta_{i,N+1} + D(x)\frac{d}{dx}\Phi(x)\big|_0\delta_{i,1} +$$

$$+\sum_{j=1}^{N+1}\Phi_j\int_0^H dxD(x)\frac{d}{dx}\Psi_i(x)\cdot\frac{d}{dx}\Psi_j(x)$$

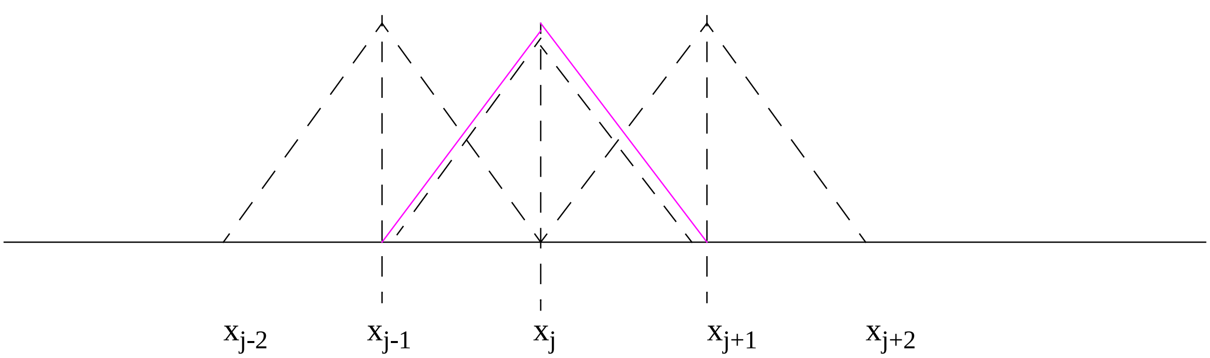


FIGURE VIII.4. Determination of inner product

$$(1) = -D(x)\frac{d}{dx}\Phi(x)\big|_H\delta_{i,N+1} + D(x)\frac{d}{dx}\Phi(x)\big|_0\delta_{i,1} +$$

$$+ \underbrace{\Phi_{i-1}D_{i-1}\int_{x_{i-1}}^{x_i}\frac{d\Psi_{i-1}}{dx}\cdot\frac{d}{dx}\Psi_i(x)dx} + \underbrace{\Phi_iD_{i-1}\int_{x_{i-1}}^{x_i}\frac{d\Psi_i}{dx}\cdot\frac{d\Psi_i}{dx}dx}$$

i-1 i-1

$$+ \Phi_i D_i \int_{x_i}^{x_{i+1}} dx \frac{d}{dx}\Psi_i(x)\frac{d}{dx}\Psi_i(x) + \Phi_{i+1} D_i \int_{x_i}^{x_{i+1}} dx \frac{d}{dx}\Psi_i(x)\frac{d\Psi_{i+1}}{dx}$$

$$(1) = -D(x)\frac{d}{dx}\Phi(x)\Big|_H + D(x)\frac{d}{dx}\Phi(x)\Big|_0 + \Phi_{i-1}D_{i-1}\int_{x_{i-1}}^{x_i}\left(-\frac{1}{h_{i-1}}\right)\cdot\left(\frac{1}{h_{i-1}}\right)dx +$$

$$+ \Phi_i D_{i-1}\int_{x_{i-1}}^{x_i} dx\left(\frac{1}{h_{i-1}}\right)^2 + \Phi_i D_i \int_{x_i}^{x_{i+1}} dx\left(\frac{1}{h_i}\right)^2 + \Phi_{i+1} D_i \int_{x_i}^{x_{i+1}}\left(-\frac{1}{h_i}\right)\left(\frac{1}{h_i}\right)dx$$

$$(1) = -D(x)\frac{d}{dx}\Phi(x)\Big|_H + D(x)\frac{d}{dx}\Phi(x)\Big|_0 +$$

$$-\frac{\Phi_{i-1}D_{i-1}}{(h_{i-1})^2}(x_i - x_{i-1}) + \frac{\Phi_i D_{i-1}}{h_{i-1}} + \frac{\Phi_i D_i}{h_i} - \frac{\Phi_{i+1}D_i}{h_i}$$

$$(1) = -D(x)\frac{d}{dx}\Phi(x)\Big|_H + D(x)\frac{d}{dx}\Phi(x)\Big|_0 + \frac{D_{i-1}}{h_{i-1}}(\Phi_i - \Phi_{i-1}) + \frac{D_i}{h_i}(\Phi_i - \Phi_{i+1})$$

VIII.3. General Formulation of Basis Functions

Consider the i-th mesh element:

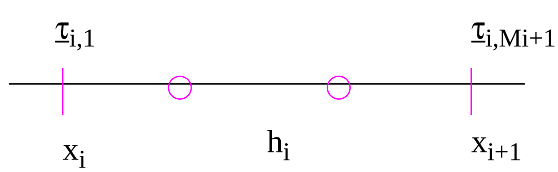
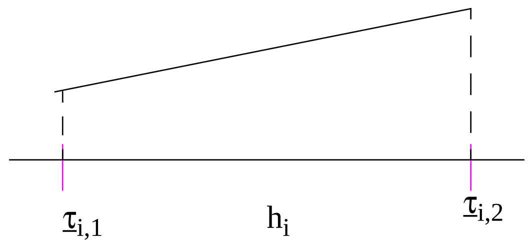


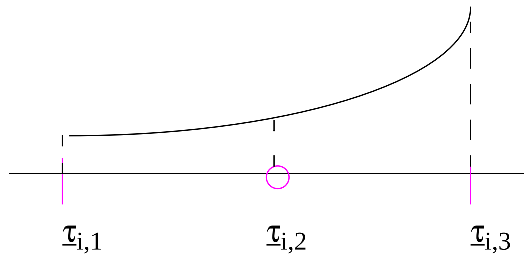
FIGURE VIII.5. I-th mesh with internal nodes

where $\tau_{i,1}$ and τ_{i,M_i+1} - two end nodes, $o(M_i - 1)$ interval node, M_i - degree of interpolating polynomial in the i-th element.

In the case of $M_i = 1$ - linear basis functions:



In the case of $M_i = 2$ - quadratic basis functions:



Then the load basis functions can be written as:

$$\Psi_{i,j}\left(\frac{\tau_{i,k} - x_i}{h_i}\right) = \delta_{k,j} = \begin{cases} 1 & , k = j \\ 0 & , k \neq j \end{cases} \qquad \text{(VIII. 15)}$$

where j is the index of local subdivision.The local interpolating polynomial can be expressed as

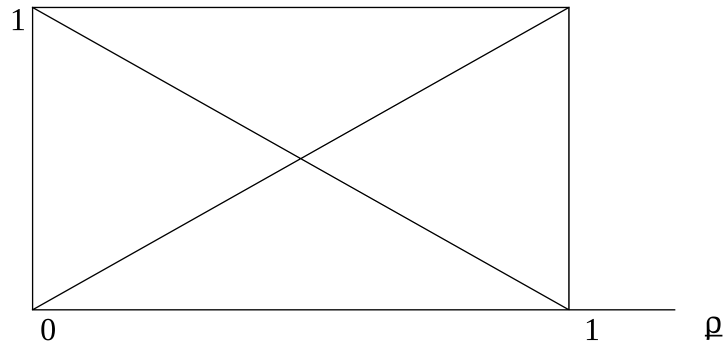
$$P_{M_i}^{(i)}(x) = \sum_{j=1}^{M_i+1} \Phi_{ij} \Psi_{i,j}\left(\frac{x - x_i}{h_i}\right) \qquad , x_i \leq x \leq x_{i+1} \qquad \text{(VIII. 16)}$$

where $\Phi_{ij} = \Phi(\tau_{ij})$. For

$$M_i = 1 \Rightarrow \Psi_{i,1}(\rho) = 1 - \rho \qquad \text{(VIII. 17)}$$

$$\Psi_{i,1}(\rho) = \rho$$

where $0 \le \rho \le 1 \qquad \rho = \frac{x - x_i}{h_i}$.

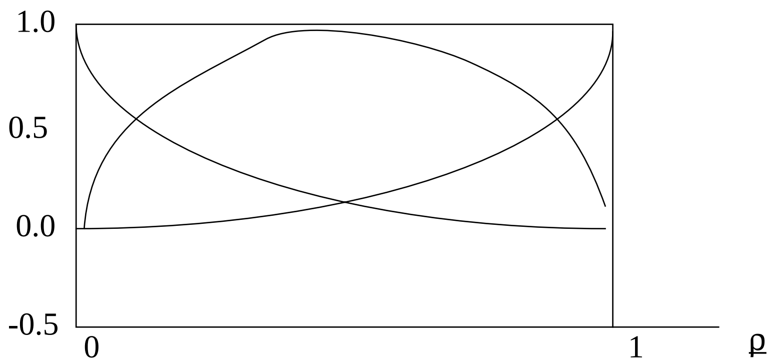


For Mi = 2, we hav

$$\Psi_{i,1}(\rho) = 1 - 3\rho + 2\rho^2 \qquad \text{(VIII. 18)}$$

$$\Psi_{i,2}(\rho) = 4\rho(1 - \rho)$$

$$\Psi_{i,3}(\rho) = \rho(2\rho - 1)$$



VIII.4. General Interpolating Polynomials

A function can be replaced by interpolatin polynomial as

$$\Phi(x) \cong \sum_{i=1}^{N} P_{M_i}^{(i)}(x) = \sum_{i=1}^{N} \sum_{j=1}^{M_i+1} \Phi_{ij} \Psi_{ij}\left(\frac{x - \tau_{i1}}{h_i}\right) \qquad \text{(VIII. 19)}$$

Thus, if all elements are linear, da basis functions in Eq. 19 are given as:

$$\Psi_i(x) = \Psi_{i,1}\left(\frac{x - x_1}{h_i}\right) \qquad i = 1$$

$$\Psi_i(x) = \Psi_{i-1,2}\left(\frac{x - x_{i-1}}{h_{i-1}}\right) + \Psi_{i,1}\left(\frac{x - x_i}{h_i}\right) \qquad ,i = 2, \ldots, N$$

$$\Psi_{N+1}(x) = \Psi_{N,2}\left(\frac{x - x_N}{h_N}\right) \qquad , i = N + 1$$

If all elements are quadratic, we introdice (2N+1) points

$$\Phi(x) = \sum_{i=1}^{2N+1} \Phi_i \Psi_i(x)$$

$$\Psi_1(x) = \Psi_{1,1}\left(\frac{x - x_1}{h_1}\right)$$

$$\Psi_i(x) = \Psi_{i/2,2}\left(\frac{x - x_{i/2}}{h_i}\right) \qquad i = 2, 4, \ldots 2N$$

$$\Psi_i(x) = \Psi_{\left(i - \frac{1}{2}\right),3}\left(\frac{x - x_{(i-1)/2}}{h(i-1)/2}\right) + \Psi_{\frac{i+1}{2},1}\left(\frac{x - x_{\frac{i+1}{2}}}{h_{\frac{i+1}{2}}}\right) \qquad i = 3, \ldots 2N$$

$$\Psi_{2N+1}(x) = \Psi_{N,3}\left(\frac{x - x_N}{h_N}\right)$$